# A Finite-Element Analysis Including Transverse Shear Effects for Applications to Laminated Plates

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**A finite-element analysis technique which includes the effects of transverse shear deformation and is readily adaptable to arbitrary laminated plates is described. The discrete element employed is a rectangle with 28 degrees-of-freedom which include extension, bending, and transverse shear deformation states. The displacement formulation is based on a refined theory for laminates which allows the deformed normal to rotate to include transverse shear deformations. Results for plate deformations and internal stress distributions, including transverse shear stresses, are shown to compare quantitatively with the theory of elasticity for selected example problems. Additional results for laminate deformation behavior are in good agreement with the shear deformation theory of Ambartsumyan. The method described can easily be incorporated into existing matrix analysis schemes which can then be used with confidence in analyzing advanced composite structures.**

## Nomenclature

$a,b$ = in-plane dimensions of rectangular plates, in.
$A_{ij}$ = in-plane constants in laminate constitutive equations, lb/in.
$B_{ij}$ = coupling constants in laminate constitutive equations, lb.
$D_{ij}$ = bending constants in laminate constitutive equations, lb-in.
$DS_{ij}$ = shear constants in equations for shear stress resultants, lb/in.
$E$ = quasi-isotropic modulus, lb/in.$^2$
$e$ = superscript denoting element $e$
$E_{11},E_{22}$ = longitudinal and transverse moduli for orthotropic material (parallel and transverse to fibers), lb/in.$^2$
$G_{12},G_{13},G_{23}$ = shear rigidities for orthotropic material with reference to principal elastic axes, lb/in.$^2$
$h$ = laminate thickness, in.
$n$ = total number of layers in laminate
$q$ = transverse uniform load, lb/in.$^2$
$q_0$ = maximum intensity of sinusoidal load, lb/in.$^2$
$Q_{ij}$ = constants in lamina constitutive equations with reference to lamina principal axes, lb/in.$^2$
$\bar{Q}_{ij}$ = constants in lamina constitutive equations with reference to laminate reference axes, lb/in.$^2$
$u^0,v^0$ = extensional displacements of laminate midplane in the $x$ and $y$ directions, in.
$u,v,w$ = displacements of any point in the laminate in the $x$, $y$, and $z$ directions, in.
$U$ = internal strain energy, lb-in.
$\alpha_i$ = constant coefficients in displacement functions
$\gamma_{xy},\gamma_{xz},\gamma_{yz}$ = shear strains at any point in the laminate
$\epsilon_x{}^0,\epsilon_y{}^0,\gamma_{xy}{}^0$ = strains of the laminate midplane
$\bar{\gamma}_x,\bar{\gamma}_y$ = "average" transverse shear strains at the laminate midplane
$\epsilon_x,\epsilon_y,\epsilon_z$ = normal strains at any point in the laminate
$\nu_{ij}$ = Poisson's ratio with reference to lamina principal elastic axes
$\sigma_x,\sigma_y,\sigma_z$ = normal stresses at any point in the laminate, lb/in.$^2$
$\theta$ = angle between lamina and laminate reference axes (positive when measured counter-clockwise from laminate to lamina axes)
$\tau_{xy},\tau_{xz},\tau_{yz}$ = shear stresses at any point in the laminate, lb/in.$^2$
$\phi_x,\phi_y$ = total rotations of sections $x$ = const and $y$ = const
$\chi_x,\chi_y,\chi_{xy}$ = components of plate curvature, in.$^{-1}$
$[b_E],[b_B],[b_S]$ = matrices relating strains and curvatures to nodal displacements
$[k]^e$ = element stiffness matrix
$[M]$ = vector of moment stress resultants $M_x$, $M_y$, and $M_{xy}$
$[N]$ = vector of in-plane stress resultants $N_x$, $N_y$, and $N_{xy}$
$[Q]$ = vector of shear stress resultants $Q_x$ and $Q_y$
$[r]$ = matrix relating nodal displacements to the displacements of any point in the laminate
$[u]$ = vector of displacements for any point in an element
$[u]^e$ = nodal displacements for element $e$

### *Subscripts*

1,2,3 = lamina reference coordinate system
$B,E,S$ = reference to bending, extension and shear, respectively
$x,y,z$ = laminate reference coordinate system

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## Introduction

RECENT developments[1–7] in the analysis of plates laminated of orthotropic materials have indicated that the thickness concept, as it is known for isotropic plates, is different for heterogeneous anisotropic plates. For such plates, the distortion of the deformed normal due to transverse shear is dependent, not only on the laminate thickness, but also on the orientation and degree of orthotropy of the individual layers. Thus, it is evident that to establish a rational laminate analysis technique which can be applied to conventional composite structures, the effects of transverse shear deformation should be included.

Ambartsumyan[1] has developed a shear deformation theory for "specially" orthotropic, symmetric laminates. The theory is based on assumed distributions of the transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$ in each layer. Whitney[2] has extended this theory to symmetric laminates with arbitrary layer orientations and has presented results for gross plate behavior (static deflections, buckling loads and natural frequencies). Comparisons with exact elasticity solutions for the cylindrical bending of cross-ply laminates given by Pagano[3] indicated the accuracy of these results. It was noted, however, that because of the assumptions made, the transverse shear stress distributions did not compare quantitatively with those of the theory of elasticity.

The investigations discussed above are useful in establishing behavioral trends for laminated plates and in defining the range of applicability of classical thin-plate theory. However, because of the complexity of the theories used, the analyses are severely limited in application and their incorporation into conventional iterative design procedures would be cumbersome if not impossible.

It is the purpose of this paper to give a brief description of a finite-element analysis technique which includes transverse shear effects and is readily adaptable to laminated anisotropic plates. Because the transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$ are derived from the equilibrium equations of elasticity for each layer, quantitative comparisons with exact elasticity solutions[3] are obtained. Results are presented for gross plate behavior which substantiate those of Whitney,[2] and additional results for internal stress distributions indicate the influence of transverse shear, especially near plate edges. It is believed that the discrete element employed can easily be incorporated into existing multipurpose computer programs which can then be used with confidence in analyzing advanced composite structures.

## General Theory

Each layer in the laminate is assumed to be in a specific three dimensional stress state so that the constitutive relations for a typical lamina $k$ with reference to the laminate coordinate axes $(x,y,z)$ can be written as

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix}^k = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0\\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45}\\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}^k \begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix}^k \quad (1)$$

where the $\bar{Q}_{ij}$ are determined in the usual manner by transforming the coefficient matrix $Q_{ij}$ in the stress-strain relations from the lamina principal elastic axes (1,2,3) to the laminate reference axes and

$$Q_{11} = E_{11}/(1 - \nu_{12}\nu_{21}),\ Q_{22} = E_{22}/(1 - \nu_{12}\nu_{21})$$
$$Q_{12} = \nu_{12}E_{22}/(1 - \nu_{12}\nu_{21})\ Q_{66} = G_{12},\ Q_{44} = G_{13},\ Q_{55} = G_{23} \quad (2)$$

The normal stress $\sigma_z{}^k$ is assumed negligible and is omitted in this development.

The relationships between the strains at any point in the laminate and the corresponding deformations are based on assumptions regarding the kinematic displacement modes for points in the laminate. In this theory, the classical Kirchoff-Love assumption for normals to the midplane is relaxed in favor of the assumed deformation mode shown in Fig. 1. The angles $\phi_x$ and $\phi_y$ are the total rotations of sections $x =$ const and $y =$ const, respectively, and are given by

$$\phi_x(x,y) = \partial w/\partial x + \bar{\gamma}_x(x,y)$$
$$\phi_y(x,y) = \partial w/\partial y + \bar{\gamma}_y(x,y) \quad (3)$$

where $\bar{\gamma}_x$ and $\bar{\gamma}_y$ are "average" transverse shear strains. It can be shown that Eqs. (3) with a shear coefficient (an estimate of section warping and a multiple of $\bar{\gamma}$) of $\frac{6}{5}$ are identical to analogous equations of the widely accepted Reissner theory.[9] They are employed in this development because, as will be shown, they can be conveniently incorporated into a finite-element displacement formulation.

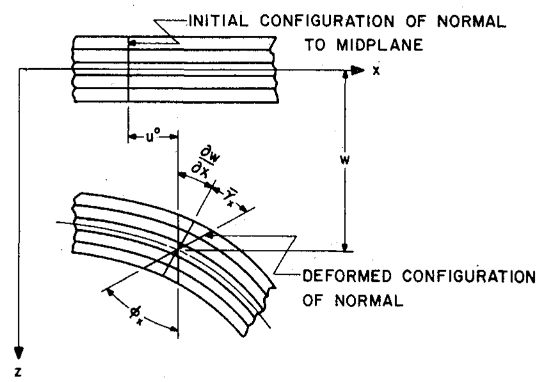


**Fig. 1 Assumed deformation mode for normals to the midplane.**

The displacements of any point in the laminate are given by

$$u(x,y,z) = u^0(x,y,0) - z\phi_x(x,y)$$
$$v(x,y,z) = v^0(x,y,0) - z\phi_y(x,y) \quad (4)$$
$$w(x,y,z) = w(x,y,0) = w(x,y)$$

The linear strain-displacement relationships for small deformations can then be written as

$$\epsilon_x = \partial u^0/\partial x - z(\partial^2 w/\partial x^2 + \partial\bar{\gamma}x/\partial x) = \epsilon_x{}^0 + z\chi_x$$
$$\epsilon_y = \partial v^0/\partial y - z(\partial^2 w/\partial y^2 + \partial\bar{\gamma}_y/\partial y) = \epsilon_y{}^0 + z\chi_y$$
$$\epsilon_z = \partial w/\partial z = 0$$
$$\gamma_{xy} = \partial u^0/\partial y + \partial v^0/\partial x - z(2\partial^2 w/\partial x\partial y + \partial\bar{\gamma}_x/\partial y + \partial\bar{\gamma}_y/\partial x) = \gamma_{xy}{}^0 + z\chi_{xy} \quad (5)$$
$$\gamma_{xz} = \partial u/\partial z + \partial w/\partial x = -\bar{\gamma}_x$$
$$\gamma_{yz} = \partial v/\partial z + \partial w/\partial y = -\bar{\gamma}_y$$

where

$$\epsilon_x{}^0 = \partial u^0/\partial x,\ \epsilon_y{}^0 = \partial v^0/\partial y,\ \gamma_{xy}{}^0 = \partial u^0/\partial y + \partial v^0/\partial x$$
$$\chi_x = -(\partial^2 w/\partial x^2 + \partial\bar{\gamma}_x/\partial x), \chi_y = -(\partial^2 w/\partial y^2 + \partial\bar{\gamma}_y/\partial y) \quad (6)$$
$$\chi_{xy} = -(2\partial^2 w/\partial x\partial y + \partial\bar{\gamma}_x/\partial y + \partial\bar{\gamma}_y/\partial x)$$

The stresses in a typical lamina $k$ can be written in terms of the midplane strains and curvatures by substituting Eqs. (5) into the constitutive Eqs. (1) to give

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}^k = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^k \left[\begin{Bmatrix}\epsilon_x{}^0\\ \epsilon_y{}^0\\ \epsilon_{xy}{}^0\end{Bmatrix} + z\begin{Bmatrix}\chi_x\\ \chi_y\\ \chi_{xy}\end{Bmatrix}\right] \quad (7)$$

$$\begin{bmatrix}\tau_{xz}\\ \tau_{yz}\end{bmatrix}^k = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}^k \begin{bmatrix}-\bar{\gamma}_x\\ -\bar{\gamma}_y\end{bmatrix} \quad (8)$$

The stress resultants are determined in the usual manner by appropriate integration of Eqs. (7) and (8) over the thickness of layer $k$ and summation over all $n$ layers of the laminate. The results are

$$\begin{bmatrix}N\\ M\end{bmatrix} = \begin{bmatrix}A & B\\ B & D\end{bmatrix}\begin{bmatrix}\epsilon^0\\ \chi\end{bmatrix} \quad (9)$$

and

$$[Q] = \begin{bmatrix}Q_x\\ Q_y\end{bmatrix} = \begin{bmatrix}DS_{44} & DS_{45}\\ DS_{45} & DS_{55}\end{bmatrix}\begin{bmatrix}-\bar{\gamma}_x\\ -\bar{\gamma}_y\end{bmatrix} = [DS]\ [\bar{\gamma}] \quad (10)$$

where

$$(A_{ij},B_{ij},D_{ij}) = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1,z,z^2)dz \qquad i,j \Rightarrow 1,2,6$$
$$DS_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}dz \qquad i,j \Rightarrow 4,5$$

Equations (9) are different from analogous equations of classical laminated plate theory since the effects of transverse

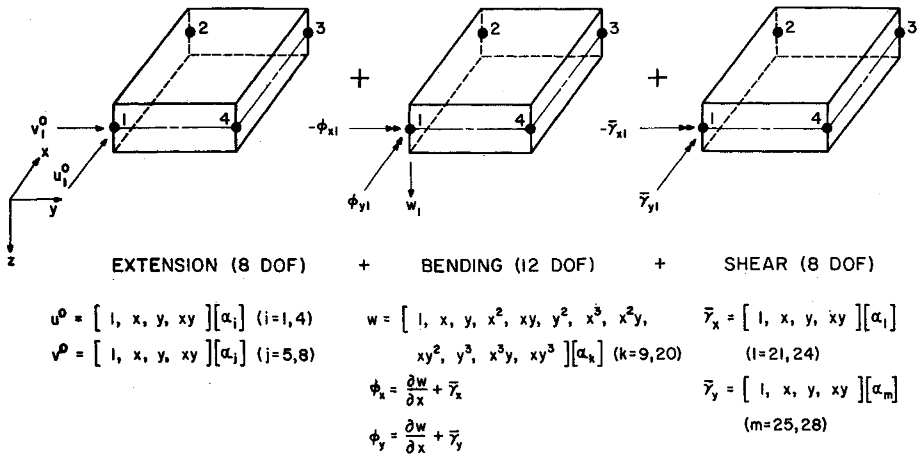


**Fig. 2 Displacement fields for rectangular plate element (28 DOF).**

shear are included in the plate curvatures $[\chi]$. Because of the assumptions made in formulating the last two of Eqs. (5), the transverse shear stresses $\tau_{xz}{}^k$ and $\tau_{yz}{}^k$ are not accurately given by Eqs. (8). That is, it is not reasonable to assume that the transverse shear strains in each lamina are equal to the "average" transverse shear strain $\bar{\gamma}$ which occurs at the laminate midplane. For this reason, the transverse shear stresses are determined using the equilibrium equations of elasticity for each layer. These are for the $i$th layer

$$\tau^i{}_{xz,z} = -(\sigma^i{}_{x,x} + \tau^i{}_{xy,y}),\ \tau^i{}_{yz,z} = -(\sigma^i{}_{y,y} + \tau^i{}_{xy,x}) \quad (11)$$

assuming that body forces are negligible. The terms on the right-hand side of the preceding equations can be determined by performing the indicated partial differentiations on Eqs. (7) for the $i$th layer. The stresses $\tau_{xz}{}^k$ and $\tau_{yz}{}^k$ can then be determined by integrating Eqs. (11) over the thickness of layer $i$ and summing from either the top or bottom surface of the laminate to layer $k$. This is done assuming that there are no shearing tractions on the top and bottom surfaces of the laminate and the stresses $\tau_{xz}$ and $\tau_{yz}$ are continuous between layers. The resulting expressions give the stresses $\tau_{xz}{}^k$ and $\tau_{yz}{}^k$ in terms of the midplane strains and curvatures. The expressions are lengthy and for brevity are omitted herein. It should be noted that because of the integration over the thickness of the laminate, Eqs. (10) should give an accurate representation of the shear stress resultants $Q_x$ and $Q_y$.

Now, the displacements Eq. (4), the midplane strains and curvatures Eq. (6), the stresses Eqs. (7) and (11), and stress resultants Eqs. (9) and (10) are given explicitly in terms of the deformations $(u^0,v^0,w,\bar{\gamma}_x,\bar{\gamma}_y)$ of the laminate midplane. Therefore, a finite-element displacement formulation can conveniently be used to completely define the elastostatic behavior of the laminate in terms of the midplane deformations.

## Finite-Element Displacement Formulation

The solution accuracy of the finite-element displacement formulation depends on the ability of assumed functions to accurately model the deformation modes of the structure. For the elastostatic response of laminated plates, in which the effects of transverse shear are included, it is thus imperative that extension, bending and shear deformation states and their interaction be modeled correctly. To accomplish this, a rectangular flat plate element shown in Fig. 2 with seven degrees-of-freedom at each nodal point is employed. The displacements of any point in the element can be written in terms of the surrounding nodal displacements by inserting the nodal coordinates into the displacement functions shown in Fig. 2 and performing a customary inversion and substitution (see for example Refs. 8 and 10) which results in

$$[u] = [r][u]^e \quad (12)$$

where $[r]$ is an interpolation matrix, $[u]^e$ are the displacements of the nodes surrounding element $e$, and

$$[u]^T = [u^0v^0w\phi_x\phi_y\bar{\gamma}_x\bar{\gamma}_y]$$

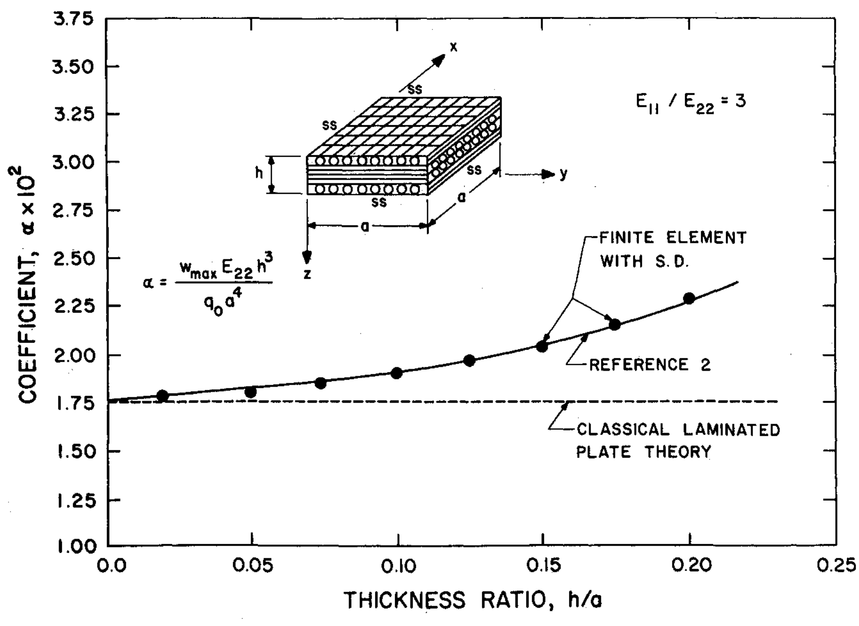


**Fig. 3 The effect of transverse shear on the deflection of a square, glass-epoxy laminate subjected to a sinusoidal load.**

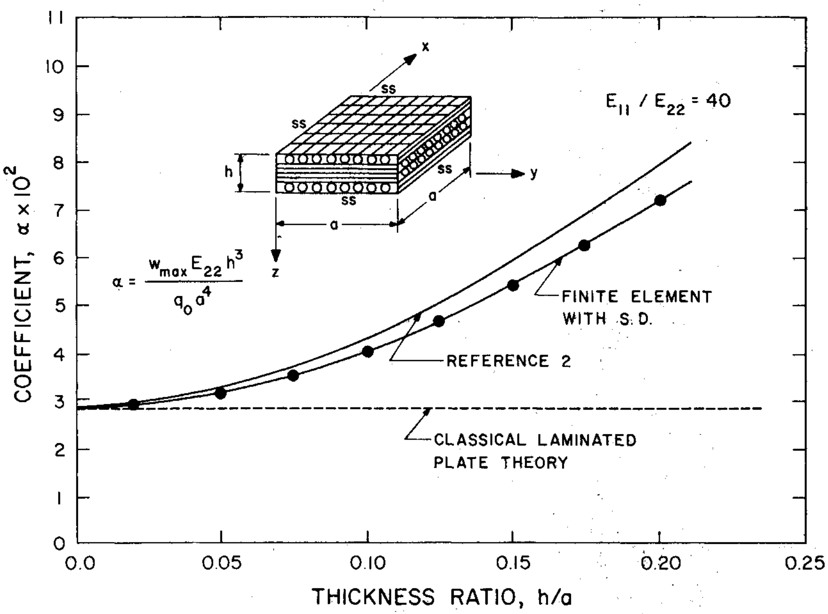


**Fig. 4 The effect of transverse shear on the deflection of a square, graphite-epoxy laminate subjected to a sinusoidal load.**

The midplane strains and curvatures can be written in terms of the nodal displacements by performing the operations indicated in Eqs. (6) on Eqs. (12) to give

$$[\epsilon^0] = [b_E][u]^e,\ [\chi] = [b_B][u]^e,\ [\bar{\gamma}] = [b_S][u]^e \quad (13)$$

The subscripts $E$, $B$, and $S$ refer to extension, bending, and shear, respectively.

The internal strain energy of the element can be determined by integrating the products of in-plane stress resultants and extensional strains, moment stress resultants and bending curvatures, and shear stress resultants and shear strains over the area of an element. The result is

$$U = \frac{1}{2}\int_A [N]^T[\epsilon^0]dA + \frac{1}{2}\int_A [M]^T[\chi]dA + \frac{1}{2}\int_A [Q]^T[\bar{\gamma}]dA \quad (14)$$

Substituting Eq. (13) and the expressions for $[N]$, $[M]$, and $[Q]$ given in Eqs. (9) and (10) into Eq. (14) results in the following expression for the internal strain energy

$$U = \tfrac{1}{2}[u]^{eT}[k]^e[u]^e \quad (15)$$

Where $[k]^e$ is the stiffness matrix for element $e$ and is given as

$$[k]^e = \int_A \{[b_E]^T[A][b_E] + [b_B]^T[B][b_E]\}dA + \int_A \{[b_E]^T[B][b_B] + [b_B]^T[D][b_B]\}dA + \int_A \{[b_S]^T[DS][b_S]\}dA \quad (16)$$

The nodal displacements can be determined by performing a stiffness analysis of the entire laminated plate structure. External loadings and boundary conditions are subject to the usual finite element analysis considerations. Once the nodal displacements are known, the complete elastostatic response of the laminate is defined.

## Numerical Examples and Discussion

To test the validity of the finite-element analysis technique and to establish its range of applicability, several numerical examples were investigated. The first of these were symmetric, cross-ply (0°/90°/90°/0°) laminates. These examples were studied for the purpose of comparing finite-element results with those of Ambartsumyan's theory[1] which was applied by Whitney.[2] Figures 3 and 4 show results for the variation of a maximum nondimensional deflection parameter with plate thickness ratio $h/a$ for square glass-epoxy and graphite-epoxy laminates. The plates were simply-supported on all edges and subjected to a sinusoidal loading of the form $q_0 \sin\pi x/a \sin\pi y/b$. The boundary conditions and material properties are described in detail in Ref. 2.

Good agreement is obtained with the results of Whitney[2] for both the glass-epoxy and the graphite-epoxy laminates. For the glass-epoxy laminate in which the modulus ratio $E_{11}/E_{22}$ is 3 for each ply, the increase in deflection due to transverse shear is 8.6% at a thickness ratio $h/a = 0.1$. For the graphite-epoxy ($E_{11}/E_{22} = 40$), the increase in deflection is 50% at a thickness ratio $h/a = 0.1$. It is apparent then, as indicated by Whitney, that the effects of transverse shear on the deflection behavior of these cross-ply laminates are considerably increased with the degree of orthotropy of the individual layers. Thus, the criteria for applying classical thin-plate theory to laminated composite plates is not as simply defined as it is for homogeneous isotropic plates.

Figure 5 shows results of two finite-element analyses for the distribution of transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$ through the thickness of the cross-ply (0°/90°/90°/0°) graphite-epoxy laminate discussed previously. The particular laminate studied had a thickness ratio $h/a = 0.2$ and was subjected to a uniform load $q$. The edges were simply-supported as in the previous case. Results of the finite element analysis which does not include shear deformations were given by Pryor.[10] The finite-element idealization used here, as in the previous case, was a 6 × 6 mesh which modeled the behavior of one quarter of the plate. The stresses were computed at the element centers which explains the fractional locations (a/24, 11a/24) shown in Fig. 5.

Considerable difference can be noted in the results of the two analyses, particular in the distribution of transverse shear stress parallel to the edges near the edges. This is also true for thick isotropic plates but the differences are not as pronounced indicating again the influence of material properties and transverse heterogeneity. Similar results based on the theory of elasticity were presented by Pagano.[5]

Figure 6 shows results for the cylindrical bending of an infinite crossply (0°/90°) laminate subjected to a sinusoidal load of the form $q_0 \sin\pi x/a$. This example was studied for the purpose of comparing finite-element results with those of elasticity solutions given by Pagano.[3] The simple-support edge conditions and material properties are described in detail in ref. 3. For the thick plate ($h/a = 0.25$) considered, the results of both finite-element analyses are in good agreement with those of the theory of elasticity for the normal stress $\sigma_x$ distribution and the deformed normal configuration.

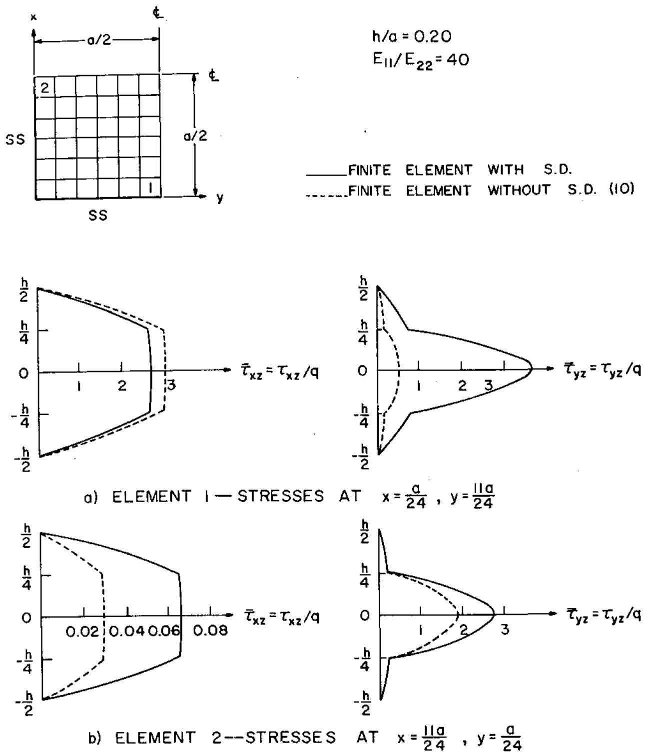


Fig. 5 Transverse shear stresses in a symmetric (0°/90°/90°/0°) graphite-epoxy laminate.

Figure 7 shows results for the cylindrical bending of a symmetric crossply (0°/90°/0°) laminate. The loading and boundary conditions are identical to those of the preceding case. For this example, both finite-element analyses give a poor approximation to the deformed normal configuration. The theory described in the preceding sections is based on the assumption that the deformed normal remains straight but not normal to the midplane. This gives a good approximation for the two layer asymmetric laminate but errs for the three layer plate where the center layer warps causing a severe discontinuity in the slope of the deformed normal at the layer interfaces.[3] This results in a maximum transverse shear stress occurring in the facing layers and not in the core. Despite the poor approximation to the deformed normal configuration, the finite element results for the stress distributions compare quantitatively with those of elasticity. Figure 7 also indicates the rapid convergence of finite element results to those of elasticity as the thickness ratio $h/a$ is decreased.

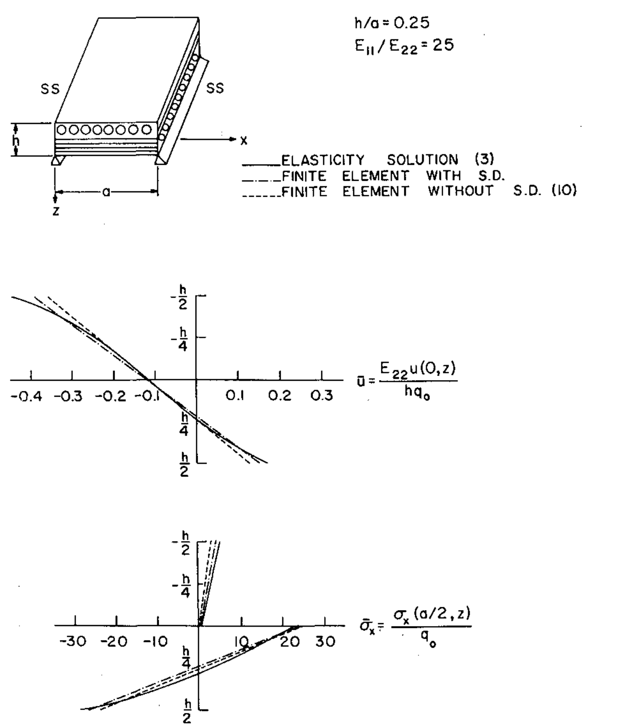


Fig. 6 Typical results for the behavior of a two-layer, graphite-epoxy, cross-ply laminate in cylindrical bending.

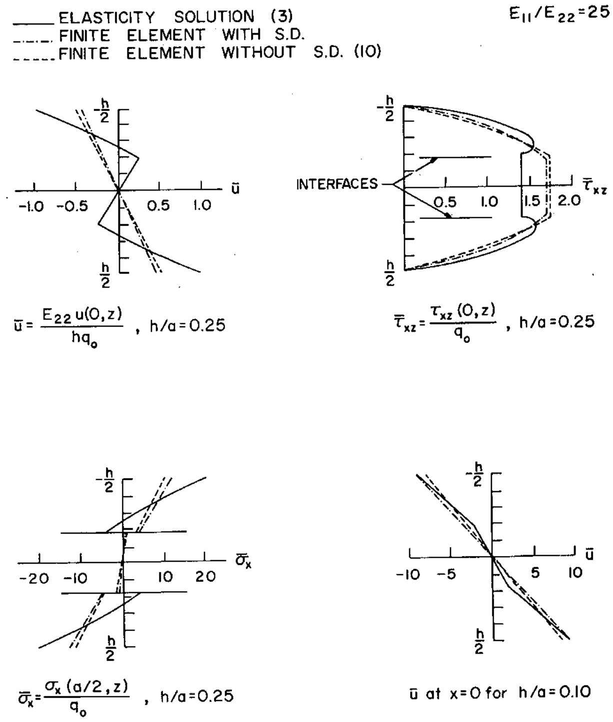


Fig. 7 Typical results for the behavior of a three-layer, graphite-epoxy, cross-ply laminate in cylindrical bending.

These results indicate that a more general approach to the analysis of the problem should include a separate angle of rotation for each layer in the laminate, instead of the present assumption of only one rotation of the midplane through the entire thickness of the plate. Such a three dimensional analysis was considered, but was deemed not feasible because of the increased computational effort required. In the present analysis, a typical area in the $xy$ plane bounded by four nodes has a total of 28 degrees-of-freedom (DOF). If an eight node three dimensional element with 24 DOF is used for each layer, a three layer laminate will require 48 DOF for the same $xy$ area. As additional layers are added to the plate, and displacement functions beyond the linear are used, the difference between the total number of DOF required becomes even greater. Undoubtedly, the three dimensional analysis would give a better representation of the deformed normal, however, the improvement in the stress distribution would be small compared to the large increase in computational effort.

## Conclusions

It has been demonstrated in this investigation that the versatile finite-element analysis technique, when based on the governing equations of a refined theory, can accurately predict the complex stress and deformation behavior of laminated anisotropic plates. It is believed that the method can conveniently be extended to the analysis of a larger class of laminates with varied loading and support conditions. Results for plate deformations and internal stress distributions are shown to compare favorably with the theory of elasticity for selected example problems. Further, results for gross plate behavior substantiate those of Ref. 2 which were obtained using a shear deformation theory developed by Ambartsumyan.[1] To the authors' knowledge, this is the only other shear deformation theory developed specifically for applications to laminated plates.

Other numerical results indicate that the criteria for applying thin-plate theory (CPT) to laminated plates is not well defined. For example, for certain laminates, the deformed normal severely warps causing the maximum transverse shear stress to be carried in the facing layers. Limits on the lamina thicknesses, the material types and layer orientations which ensure that this undesirable effect does not occur need to be established.

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# Buckling of Stiffened Multilayered Circular Cylindrical Shells with Different Orthotropic Moduli in Tension and Compression

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**An exact buckling criterion, within the framework of classical buckling theory, is derived for eccentrically stiffened multilayered circular cylindrical shells made of materials that have different orthotropic moduli in tension and compression. Such behavior is typical of many current composite materials. The buckling criterion is valid for arbitrary combinations of axial and circumferential loading, including axial compression and internal pressure as well as axial tension and lateral pressure. The material model (stress-strain relationship) involves a bilinear stress-strain curve that has a discontinuity in slope (modulus) at the origin. A numerical example of buckling of a ring-stiffened two-layered circular cylindrical shell is given to illustrate application of the buckling criterion.**

## Nomenclature†

$a$ = ring spacing (Fig. 3)
$A$ = cross-sectional area of a stiffener
$A_{ij}$ = coefficients in buckling criterion Eq. (30)
$b$ = stringer spacing (Fig. 3)
$B_{ij}$ = extensional stiffnesses of the layered shell
$C_{ij}$ = coupling stiffnesses of the layered shell
$D_{ij}$ = bending stiffnesses of the layered shell
$E$ = Young's modulus of an isotropic material
$E_x, E_y$ = Young's moduli in $x$ and $y$ directions, respectively
$E_{45}$ = Young's modulus at 45° to principal axes of orthotropy
$G$ = shearing modulus, $E/[2(1 + \nu)]$, of a stiffener
$I$ = moment of inertia of a stiffener about its centroid
$J$ = torsional constant of a stiffener
$k_x, k_y$ = weighting factors in compliance matrix Eq. (5)
$K_{ij}$ = stiffnesses in stress-strain relations Eq. (7)
$L$ = length of circular cylindrical shell (Fig. 3)
$m$ = number of axial buckle halfwaves
$\delta M_x, \delta M_y, \delta M_{xy}, \delta M_{yx}$ = variations in moments per unit length during buckling
$n$ = number of circumferential buckle waves
$N$ = number of layers in multilayered shell
$\delta N_x, \delta N_y, \delta N_{xy}$ = variations in in-plane forces per unit length during buckling
$\bar{N}_x, \bar{N}_y$ = applied axial and circumferential forces per unit length
$p$ = lateral pressure
$P$ = axial load
$R$ = radius to shell reference surface (Fig. 3)
$S_{ij}$ = compliances in strain-stress relations Eq. (2)
$t_k$ = thickness of $k$th shell layer
$\delta u, \delta v, \delta w$ = variations of axial, circumferential, and radial displacements during buckling from a membrane prebuckled shape
$x, y, z$ = axial, circumferential, and radial coordinates on shell reference surface (Fig. 3)
$\bar{z}$ = distance from stiffener centroid to shell reference surface (Fig. 3), positive when stiffener on outside
$\epsilon_x, \epsilon_y, \gamma_{xy}$ = in-plane axial, circumferential, and shear strains
$\delta\epsilon_x, \delta\epsilon_y, \delta\gamma_{xy}$ = variations in $\epsilon_x$, $\epsilon_y$, and $\gamma_{xy}$ during buckling

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† A comma indicates partial differentiation with respect to the subscript following the comma. The prefix $\delta$ denotes the variation during buckling of the symbol which follows.